Complex Variables
By Owen Fuller
Based on the notes by R. Gratwick

Introduction

A complex number has the form $z = x + iy$, where $z \in \mathbb{C}$ is a field.
Modulus of complex number (Mod): $r = \sqrt{x^2 + y^2}$
Argument of complex number (arg): $arg(z) = \{\theta : z = re^{i\theta}\} = \{Arg(z) + 2\pi k : k \in \mathbb{Z}\}$
Where $-\pi < Arg(z) \leq \pi$ is the principal value of the argument.
De Moivre's Formula: $cos(n\theta) + isin(n\theta) = (cos\theta + isin\theta)^n$
Triangle Inequality: $|z + w| \leq |z| + |w|$
Reverse Triangle Inequality: $||z| - |w|| \leq |z - w|$
Definition 1.2.1: Let $z_0 \in \mathbb{C}$, and $\varepsilon > 0$

- The open ε-disc centred at z_0 is the set

$$D_\varepsilon(z_0) = \{z \in C : |z - z_0| < \varepsilon\}$$

- The closed ε-disc centred at z_0 is the set

$$\bar{D_\varepsilon}(z_0) = \{z \in C : |z - z_0| \leq \varepsilon\}$$

- The punctured ε-disc centred at z_0 is the set

$$D'_\varepsilon(z_0) = \{z \in C : 0 < |z - z_0| < \varepsilon\} = D_\varepsilon(z_0) \backslash \{z_0\}$$

Complex Functions

Complex Functions: Consider $f : \mathbb{C} \to \mathbb{C}$. Then for each z, $f(z) = f(x + iy) = u(x, y) + iv(x, y)$.
Continuous: Let $S \subseteq \mathbb{C}$, $f : S \to \mathbb{C}$, and $z_0 \in S$. Then f is continuous at z_0 if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z_0) - f(z)| < \varepsilon \text{ whenever } z \in S \text{ satisfies } |z_0 - z| < \delta$$

Lemmas 1.3.7/8 Let $f : \mathbb{C} \to \mathbb{C}$. Then f is continuous at z_0 iff u and v are. And f is continuous iff the preimage $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\}$ is open for all open $U \subseteq \mathbb{C}$
Differentiability: Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0 and $f : U \to \mathbb{C}$. Then f is differential at z_0 if the limit below exists.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Holomorphic: Let $z_0 \in \mathbb{C}$. We say f is holomorphic at z_0 if there exists a neighbourhood U of z_0 on which f is defined and differentiable. If it's holomorphic at every $z \in U$ then f is holomorphic on U.

Chain Rule: Let $z_0 \in \mathbb{C}$, U be a neighbourhood of z_0, $g : U \to \mathbb{C}$ be such that $g(U)$ is a neighbourhood of $g(z_0)$, and $f : g(U) \to \mathbb{C}$. Suppose g is differentiable at z_0 and f is differentiable at $g(z_0)$. Then the composition of $f \circ g : U \to \mathbb{C}$ is differentiable at z_0

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0)$$

Cauchy-Riemann equations: Let f be differential at z_0, then

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \text{ and } \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

If u and v are continuously differentiable on a neighbourhood of (x_0, y_0), and satisfy C-R at (x_0, y_0), then f is differentiable at z_0.
Harmonic Functions: Let $h : \mathbb{R}^2 \to \mathbb{R}^2$. Then h is harmonic if for all $(x, y) \in \mathbb{R}^2$ we have:

$$\frac{\partial^2 h}{\partial x^2}(x, y) + \frac{\partial^2 h}{\partial y^2}(x, y) = 0$$

The Complex Exp and Log

$$\exp(z + 2\pi i) = \exp(z)$$

$\log(z) = \ln|z| + i\arg(z) = \{\ln(r) + i\theta + i2\pi k : k \in \mathbb{Z}\}$
Branch Cut:

$$L_{z_0,\phi} = \left\{z \in \mathbb{C} : z = z_0 + re^{i\phi} \text{ for } r \geq 0\right\}$$

Cut Plane: $D_{z_0,\phi}$ denotes the cut plane with a branch point at z_0 and a branch cut along $L_{z_0,\phi}$. i.e $D_{z_0,\phi} = \mathbb{C} \backslash L_{z_0,\phi}$.
Principal Branch: $\mathrm{Log}(z) := \ln|z| + i\,\mathrm{Arg}(z)$
Branch of log: $\mathrm{Log}_\phi(z) = \ln|z| + i\mathrm{Arg}_\phi(z)$

$$\text{with } \phi < \mathrm{Arg}_\phi(z) \leq \phi + 2\pi$$

Which is holomorphic on the cut plane D_ϕ, with $\mathrm{Log}'_\phi(z) = \frac{1}{z}$ for all z in D_ϕ.

Conformal Maps and Möbius Transformations

Conformal: We say $f : U \to \mathbb{C}$ is conformal if f preserves angles: i.e if the angle between the images under f of two straight lines in U is equal to the angle between the two straight lines themselves. If f is holomorphic then is preserves angles at every $z_0 \in U$ where $f'(z_0) \neq 0$.
Theorem: If $|f'(z)| \neq 0$ for all $z \in \mathbb{C}$ then $f(z)$ is conformal on $\mathbb{C}$.

Möbius Transformation: is a function of the form:

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ are such that $ad \neq bc$.
Lemma 2.2.3: To a complex matrix $M = ((a, b), (c, d))$ with $det(M) = 1$ we associate the Möbius transformation $f_M(z) = \frac{az+b}{cz+d}$. Under this correspondence we have

$$f_{M_1 M_2} = f_{M_1} \circ f_{M_2} \text{ and } f_{M^{-1}} = f_M^{-1}$$

Types of Möbius Boys:

- Translation: $f(z) = z + b$
- Rotation: $f(z) = az$, with $|a| = 1$, so that $a = e^{i\theta}$
- Dilation: $f(z) = rz$, where $r > 0$
- Inversion: $f(z) = 1/z$

Theorem 2.4.2: Let f be a Möbius transformation. Then f is a composition of a finite number of translations, rotations, dilations and, iff f does not fix the point at infinity, one inversion.
Cor 2.4.3: Möbius transformations map circlines to circlines. (circle or straight line)

Integration

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Let $[a, b] \subseteq \mathbb{R}$ be an interval, and let $f, g : [a, b] \to \mathbb{C}$ be integrable, and $\alpha\beta \in \mathbb{C}$. Then

- $\alpha f + \beta g$ is integrable
- If f is continuous and $f = \frac{dF}{dt}$ for a differentiable function $F : [a, b] \to \mathbb{C}$ Then:

$$\int_a^b f(t)dt = F(b) - F(a)$$

- $$| \int_a^b f(t)dt | \leq \int_a^b |f(t)|dt$$

Contour Integrals

Definition 3.2.3: Let $z_0, z_1 \in \mathbb{C}$ be distinct, Γ be a regular curve connecting z_0 and z_1 and $f : \Gamma \to \mathbb{C}$ be continuous. Then we define the integral of f along Γ by:

$$\int_\Gamma f(z)dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt$$

Arclength: Let Γ be a regular curve in $\mathbb{C}$. We define arclength $\ell(\Gamma)$ by

$$\ell(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}dt$$

e.g. let Γ be an arc of a circle of radius r traced through an angle θ. Then $\ell(\Gamma) = r\theta$

Lemma 3.2.1: Let Γ be parametrised by $\gamma : [0, 1] \to \mathbb{C}$, then $-\Gamma$ runs in the opposite direction, but along the same path, parametrised by $\bar{\gamma}(t) = \gamma(1 - t)$ with

$$\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz$$

M-L Lemma: Let Γ be a regular curve in $\mathbb{C}$, and let $f : \Gamma \to \mathbb{C}$ be continuous, then:

$$|\int_\Gamma f(z)dz| \leq \max_{z\in\Gamma} |f(z)|\ell(\Gamma)$$

Domain Definition: We say $D \subseteq \mathbb{C}$ is a domain if D is open and every two points in D can be connected by a contour which lies wholly in D.

Fundamental Theorem of Calculus: Let $D \subseteq \mathbb{C}$ be a domain, Γ be a contour in D joining points $z_0, z_1 \in D$, and $f : D \to \mathbb{C}$ have an antiderivative F on D Then:

$$\int_\Gamma f(z)dz = F(z_1) - F(z_0)$$

Path Independence: Let $D \subseteq \mathbb{C}$ be a domain, and $f : D \to \mathbb{C}$ be continuous. Then the following are equivalent:

- f has an antiderivative F on D
- $\int_\Gamma f(z)dz = 0$ for all closed contours Γ in D
- All contour integrals $\int_\Gamma f(z)dz$ are independent of path Γ, and depend only on endpoints

Cauchy's Integral Theorem: Let Γ be a loop, and f be holomorphic inside and on Γ. Then:

$$\int_\Gamma f(z)dz = 0$$

Theorem 3.4.11: Let Γ be a loop which does not pass through z_0. Then:

$$\int_\Gamma \frac{1}{z - z_0} = \begin{cases} 2\pi i & \text{if } z_0 \in \text{Int}(\Gamma) \\ 0 & \text{otherwise} \end{cases}$$

Cauchy Integral Formula: Let Γ be a loop, z_0 be in the interior of Γ, and f be holomorphic inside and on Γ. Then:

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz$$

Generalised Cauchy Integral Formula: Let Γ be a loop, f be holomorphic inside and on Γ, and z lie inside Γ. Then f is infinitely differentiable at z and for all positive integers n:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z)^{n+1}} dw$$

Liouville's Theorem and Applications

Liouville's Theorem: Let f be holomorphic $\mathbb{C}$ and be bounded, i.e. satisfy for some $M > 0$, $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is constant.

Fundamental Theorem of Algebra: Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. Then if P is non-constant, P has at least one root, i.e. there exists at least one z st $P(z) = 0$.

The Maximum Modulus Principle

Theorem 3.7.1: Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$ and $R > 0$ be such that the closed disc $\bar{D}_R(z_0) \subseteq D$, and f be holomorphic on D Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})dt$$

Maximum Modulus Principle: Let $D \subseteq \mathbb{C}$ be a domain, and f be holomorphic and bounded on D, $|f(z)| \leq M$, say, for all $z \in D$, for some $M > 0$. If $|f(z)|$ achieves its maximum at $z_0 \in D$, then f is constant on D.

Maximum/Minimum Principle for Harmonic Functions: Let $D \subseteq \mathbb{R}^2$ be a domain, and $\phi : D \to \mathbb{R}$ be harmonic, such that ϕ is bounded above or below on D by $M > 0$, and $\phi(z_0) = M$ for some $z_0 \in D$. Then ϕ is constant on D.

Series

Infinite Series

Definition 4.1.1: We say a series converges if the sequence $S_n \in \mathbb{C}$ of partial sums $S_n = \Sigma_{j=0}^n z_j$ is a convergent sequence, with limit $S \in \mathbb{C}$, in which case we say that $\Sigma_{j=0}^n z_j = S$. Otherwise the series is divergent.

If S_n is a convergent series, then $z_n \to 0$ as $n \to \infty$.

While $z_n \to 0$ is a necessary condition for the series to converge, it is not sufficient. e.g. $\Sigma_{j=1}^n \frac{1}{j}$ is divergent.

The Comparison Test: Let $z_n \in \mathbb{C}$ be a sequence such that $|z_n| \leq M_n$ for some non-negative real numbers M_n, for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$, where $\Sigma_{j=0}^n M_j$ is a convergent series. Then $\Sigma_{j=0}^n z_j$ is a convergent series.

Lemma 4.1.7: $\Sigma_{j=0}^n c^j$ is convergent iff $|c| < 1$.

The Ratio Test: Let $z_n \in \mathbb{C}$ be a sequence, and suppose that

$$\lim_{n\to\infty} |\frac{z_n + 1}{z_n}| = L$$

- If $L < 1$, the series $\Sigma_{j=0}^n z_j$ is convergent
- If $L > 1$, the series $\Sigma_{j=0}^n z_j$ is divergent
- If $L = 1$ then the test is inconclusive

Pointwise Convergence: Let $S \subseteq \mathbb{C}$, and $f_n : S \to \mathbb{C}$ be a sequence of functions. We say that f_n converges pointwise to a function $f : S \to \mathbb{C}$ if for each $z \in S$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$|f_n(z) - f(z)| < \varepsilon \text{ whenever } n \geq N$$

Thus the sequence of complex numbers defined by $f_n(z)$ converges to $f(z)$

Uniform Convergence: Let $S \subseteq \mathbb{C}$, and $f_n : S \to \mathbb{C}$ be a sequence of functions. We say that f_n converges uniformly to a function $f : S \to \mathbb{C}$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $z \in S$:

$$|f_n(z) - f(z)| < \varepsilon \text{ whenever } n \geq N$$

Thus the sequence of complex numbers defined by $f_n(z)$ converges to $f(z)$, but moreover converges, roughly speaking, at the same rate,

Weierstrass M-test: Let $S \subseteq \mathbb{C}$, $f_n : S \to \mathbb{C}$ be a sequence of functions, $M \geq 0$ be a sequence of non-negative numbers, such that for all $z \in S$ and all $n \geq n_0$, for some $n_0 \in \mathbb{N}$, we have that $|f_n(z)| \leq M_n$, and the series $\Sigma_{j=0}^\infty M_j$ converges. Then the series $\Sigma_{j=0}^\infty f_j(z)$ converges uniformly on S.

Lemma 4.1.21: Let $S \subseteq \mathbb{C}$, $f_n : S \to \mathbb{C}$ be continuous functions which converge uniformly to a function $f :\to \mathbb{C}$, and Γ be a contour inside S. Then the sequence of complex numbers $\int_\Gamma f_n(z)dz$ converges to $\int_\Gamma f(z)dz$.

Lemma 4.1.22: Let $S \subseteq \mathbb{C}$, $f_n : S \to \mathbb{C}$ be continuous functions such that the series $\Sigma_{j=0}^\infty f_j(z)$ converges uniformly on S, and Γ be a contour inside S. Then

$$\int_\Gamma \sum_{j=0}^\infty f_j(z)dz = \sum_{j=0}^\infty \int_\Gamma f_j(z)dz$$

Theorem 4.1.23: Let $D \subseteq \mathbb{C}$ be a simply connected domain, and f_n be holomorphic on D and converge uniformly to a function $f : D \to \mathbb{C}$. Then f is holomorphic on D.

Power Series

Power Series Definition: Let $z_0 \in \mathbb{C}$ and $a_n \in \mathbb{C}$ be a sequence. A power series is an infinite series in the form.

$$\sum_{j=0}^{\infty} a_j(z - z_0)^j$$

Where a_j are the coefficients of the power series.
Theorem 4.2.2: Let $\Sigma_{j=0}^{\infty} a_j(z - z_0)^j$ be a power series. Then there is a number $R \in [0, \infty) \cup \{\infty\}$, such that

- The series converges on $D_R(z_0)$
- The series converges uniformly on $\bar{D}_r(z_0)$ for any $r \in [0, R)$
- The series diverges on $\mathbb{C}\backslash\bar{D}_R(z_0)$

Where R is the radius of convergence of the power series. If the sequence $|\frac{a_j}{a_{j+1}}|$ has a limit, then the radius of convergence R is equal to this limit.
Theorem 4.2.6: Let $f(z) = \Sigma_{j=0}^{\infty} a_j(z - z_0)^j$ be a power series, with radius of convergence R. Then f is holomorphic on $D_R(z_0)$.

Taylor Series

Taylor Series Definition: Let $z_0 \in \mathbb{C}$ and f be holomorphic at z_0. The Taylor Series of f centred at z_0 is the power series:

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

Theorem 4.3.2: Let $z_0 \in \mathbb{C}$, $R > 0$, and suppose f is holomorphic on $D_R(z_0)$. Then the Taylor series for f centred at z_0 converges to $f(z)$ for all $z \in D_R(z_0)$, and the convergence is uniform on $\bar{D}_r(z_0)$ for all $0 \leq r < R$.
Analytic: Let $U \subseteq \mathbb{C}$ be open, and $f : U \to \mathbb{C}$. Then f is analytic if at every point $z \in U$, f can be expressed as a convergent power series.
Theorem 4.3.5: Let $U \subseteq \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ be holomorphic. Then f is analytic.
Examples: $\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}$
$\cos(z) = \sum_{j=0}^{\infty}(-1)^j \frac{z^{2j}}{(2j)!}$
$\sin(z) = \sum_{j=0}^{\infty}(-1)^j \frac{z^{2j+1}}{(2j+1)!}$
Lemma 4.3.10: Let $z_0 \in \mathbb{C}$, $R > 0$, $\alpha, \beta \in \mathbb{C}$, and f, g be holomorphic on $D_R(z_0)$. Then

- $$f'(z) = \sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!}(z - z_0)^j$$
  For $z \in D_R(z_0)$. i.e. the Taylor series for f' is found by differentiating the Taylor series for f term-by-term.
- The Taylor series for $\alpha f + \beta g$ centred at z_0, valid on $D_R(z_0)$, is the series:
  $$\sum_{j=0}^{\infty} \frac{\alpha f^{(j)}(z_0) + \beta g^{(j)}(z_0)}{j!}(z - z_0)^j$$
- The Taylor series for fg centred at z_0, valid on $D_R(z_0)$, is the series:
  $$\sum_{j=0}^{\infty} \frac{1}{j!}\left(\sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0)g^{(j-k)}(z_0)\right)(z - z_0)^j$$

Laurent Series

Laurent Series Definition: Let $z_0 \in \mathbb{C}$, and $..., a_{-1}, a_0, a_1, ...$ be a doubly-infinite sequence of complex numbers. A Laurent series centred at z_0 has the form:

$$\sum_{j=0}^{\infty} a_j(z - z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z - z_0)^{-j} = \sum_{j=-\infty}^{\infty} a_j(z - z_0)^j$$

Which converges if each of the two series on the LHS converge. i.e the Laurent series converges for values of $z \in \mathbb{C}$ such that $r < |z - z_0| < R$.
Annulus: Let $z_0 \in \mathbb{C}$, and $r, R \in [0, \infty) \cup \{\infty\}$ Then:

- The open annulus of radii r and R centred at z_0 is the set
  $$A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$$
- The closed annulus of radii r and R centred at z_0 is the set
  $$\bar{A}_{r,R}(z_0) = \{z \in \mathbb{C} : r \leq |z - z_0| \leq R\}$$

Thus a Laurent series converges on an annulus.
Theorem 4.4.4: Let $0 \leq r < R \leq \infty$ and f be holomorphic on $A_{r,R}(z_0)$. Then f can be expressed as a Laurent series centred at z_0 which converges on $A_{r,R}(z_0)$, uniformly on $\bar{A}_{r_1,R_1}(z_0)$ where $r < r_1 \leq R_1 < R$. Moreover the coefficients of the Laurent series are given by:

$$a_j = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{(z - z_0)^{j+1}}dz$$

For any loop Γ lying inside $A_{r,R}(z_0)$ and containing z_0 in its interior.

Uniqueness of Laurent Series: Let $z_0 \in \mathbb{C}$ and $0 \leq r < R \leq \infty$, and suppose the series $\Sigma_{j=-\infty}^{\infty} c_j(z - z_0)^j$ converges on the annulus $A_{r,R}(z_0)$. Then there is a function f which is holomorphic on $A_{r,R}(z_0)$ with Laurent series centred at z_0 given by

$$f(z) = \sum_{j=-\infty}^{\infty} c_j(z - z_0)^j$$

Singularities and Zeros

Singularity: Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in \mathbb{C}$ and $f : D \to \mathbb{C}$ be a function. We say z_0 is a singularity of f if f is not holomorphic at z_0. A singularity is isolated if there exists $R > 0$ such that f is holomorphic on the punctured disc $D'_R(z_0)$ centred at z_0.
Zeros: Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0 and f be holomorphic on U. Then z_0 is a zero of f is $f(z_0) = 0$. z_0 is a zero of finite order if there exists a positive integer m such that:

$$f(z_0) = f'(z_0) = ... = f^{(m-1)}(z_0) = 0 \text{ but } f^{(m)}(z_0) \neq 0$$

Where z_0 here is a zero of order m. If $m = 1$ then the zero is a simple zero. A zero z_0 of f is isolated if there exists $R > 0$ such that $f(z) \neq 0$ for $z \in D'_R(z_0)$.
Proposition 4.5.4: Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and f be holomorphic on U, with a zero of finite order at z_0. Then z_0 is isolated.
Corollary 4.5.6: Let $z_0 \in \mathbb{C}$ be a singularity of a rational function $f = P/Q$. Then z_0 is isolated.
Singularity of a Holomorphic Function: Let $z_0 \in \mathbb{C}$ be an isolated singularity of a function f which is holomorphic on $D'_R(z_0)$ for some $R > 0$. Then f has a Laurent expansion centred at z_0 that is valid on $A_{0,R}(z_0)$. Suppose $f(z) = \Sigma_{j=-\infty}^{\infty} a_j(z - z_0)^j$ is the Laurent expansion centred at z_0, valid on $A_{0,R}(z_0)$. Then z_0 is:

- **Removable Singularity:** of f if $a_j = 0 \; \forall \; j < 0$, i.e.
  $$f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j$$
- **Pole of Order** m**:** of f for a positive integer m, if $a_j = 0$ for $j < -m$ and $a_{-m} \neq 0$, i.e.
  $$f(z) = \sum_{j=-m}^{\infty} a_j(z - z_0)^j$$
- **Essential Singularity:** if $a_j \neq 0$ for infinitely many negative values of j.

Theorem 4.5.8: Let $z_0 \in \mathbb{C}$ be a removable singularity of a function which is holomorphic on $D'_R(z_0)$ for some $R > 0$. Then $f(z_0)$ can be redefined so that f is holomorphic at z_0.

Lemma 4.5.11: Let f, g be holomorphic at z_0, where z_0 is a zero of g of order m. Then:

- If z_0 is not a zero of f, then f/g has a pole of order m at z_0
- If z_0 is a zero of order k of f, then f/g has a pole of order $m-k$ at z_0 if $m > k$, and has a removable singularity at z_0 otherwise.

Analytic Continuation

Analytic Continuation: Let $D \subseteq \tilde{D} \subseteq \mathbb{C}$ be domains, and $f : D \to \mathbb{C}$ be holomorphic. We say that a holomorphic function $F : \tilde{D} \to \mathbb{C}$ is an analytic continuation of f if $F(z) = f(z)$ for $z \in D$.

Identity Theorem: Let $D \subseteq \mathbb{C}$ be a domain $z_0 \in D$, f be holomorphic on D and such that $f(z) = 0 \ \forall z \in D_R(z_0)$, for some $R > 0$. then $f(z) = 0$ for all $z \in D$. This applies if '0' is replaced with holomorphic on D $g(z)$ also.

Corollary 4.6.7: Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$, and f be holomorphic on D and such that $f(z_n) = 0$ for a sequence of distinct points $z_n \in D$ which converges to z_0. Then $f(z) = 0$ for all $z \in D$. **Corollary 4.6.8:** Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$, and f, g be holomorphic on D such that $f(z_n) = g(z_n)$ for a sequence of distinct points $z_n \in D$ which converge to z_0. Then $f(z) = g(z)$.

Example: Consider two holomorphic functions $f(z) = \cos^2(z) + \sin^2(z)$, and $g(z) = 1$. Since we know that $\cos^2 x + \sin^2 x = 1$ for $x \in \mathbb{R}$, we know that $f = g$ on the real axis. Since this contains convergent sequences, e.g. $1/n \to 0$, they agree on their common domain, which is the whole complex plane. so $\cos^2(z) + \sin^2(z) = 1$ for all $z \in \mathbb{C}$.

The Residue Calculus

Theorem 5.1.1: Let $z_0 \in \mathbb{C}$, f be holomorphic on the punctured disc $D'_R(z_0)$ for some $R > 0$, with an isolated singularity at z_0, and Γ be a loop inside $D'_R(z_0)$, with z_0 in its interior. Then:

$$\int_\Gamma f(z)dz = 2\pi i a_{-1}$$

Where a_{-1} is the coefficient of the $(z-z_0)^{-1}$ term in the Laurent expansion of f centred at z_0 valid on $D'_R(z_0)$

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

Residue: Let $z_0 \in \mathbb{C}$, and f be holomorphic on the punctured disc $D'_R(z_0)$, for some $R > 0$, with an isolated singularity at z_0. Then the residue of f at z_0 is

$$\operatorname{Res}(f, z_0) = a_{-1}$$

Where the Laurent series of f valid on $D'_R(z_0)$ is:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

Lemma 5.1.4: Let $z_0 \in \mathbb{C}$, and f be holomorphic on the punctured disc $D'_R(z_0)$ for some $R > 0$, with a removable singularity at z_0. Then $\operatorname{Res}(f, z_0) = 0$.

Lemma 5.1.5: Let $z_0 \in \mathbb{C}$, and f be holomorphic on the punctured disc $D'_R(z_0)$ for some $R > 0$, with a pole of order m at z_0. Then:

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right)$$

Lemma 5.1.7: Let g, h be holomorphic on $D'_R(z_0)$, for some $R > 0$ such that h has a simple zero at z_0, while $g(z_0) \neq 0$. Then defining $f = g/h$, we have that

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

Example 5.1.8:

Cauchy Residue Theorem: Let Γ be a loop, and f be holomorphic inside and on Γ except for finitely many isolated singularities $z_1, ..., z_k$ in the interior of Γ, then:

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j)$$

The Argument Principle

Meromorphic: Let $D \subseteq \mathbb{C}$ be a domain. A function f is meromorphic on D if for all $z \in D$, either f has a pole of some finite order at z or f is holomorphic at z.

Lemma 5.2.2: Let $D \subseteq \mathbb{C}$ be a domain, Γ be a loop in D, and f be meromorphic on D, and not identically zero. Then f has a finite number of poles and zeros on the interior of Γ.

Zeros of Meromorphic Functions: Let Γ be a loop, and f be meromorphic on the interior of Γ, with zeros $w_1, ..., w_l$ and poles $z_1, ..., z_k$ in the interior of Γ. Then $N_0(f)$, the number of zeros of f inside Γ, counted with multiplicity, and $N_\infty(f)$, the number of poles of f inside Γ, counted with multiplicity defined as:

$$N_0(f) = \sum_{j=1}^{l} \text{order of } w_j, \quad \text{and} \quad N_\infty(f) = \sum_{j=1}^{k} \text{order of } z_j$$

The Argument Principle: Let Γ be a loop in $\mathbb{C}$, and f be meromorphic on the interior of Γ, and holomorphic and non-zero on Γ, then:

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = N_0(f) - N_\infty(f)$$

Corollary 5.2.6: !

Rouché's Theorem: Let Γ be a loop, and f, g be holomorphic on and inside Γ such that for all $z \in \Gamma$,

$$|f(z) - g(z)| < |f(z)|$$

Then $N_0(f) = N_0(g)$

Fundamental Theorem of Algebra: Let $g(z) = a_n z^n + ... + a_1 z + a_0$ be a polynomial of degree n, for $a_0, ..., a_n \in \mathbb{C}$. Then g has n zeros, counted with multiplicity.

Open Mapping Theorem: Let $D \subseteq \mathbb{C}$ be a domain and suppose f is non-constant and holomorphic on D. Then the image of D under f, $f(D) = \{f(z) : z \in D\}$, is an open subset of $\mathbb{C}$.

Maximum Modulus Principle: Let $D \subseteq \mathbb{C}$ be a domain and f be holomorphic and non-constant on D. Then $|f(z)|$ does not attain a maximum on D.

Trigonometric Integrals

We can now start applying residue calculus to evaluate real integrals. Integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta$$

for a rational function R can often be evaluated by considering a contour integral of an appropriate function around the unit circle. We have that:

$$\cos\theta = \operatorname{Re}(z) = \frac{z + \overline{z}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \operatorname{Im}(z) = \frac{z - \overline{z}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

!

Improper Integrals

Improper Integrals: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then we define the improper integrals whenever the appropriate limit exists.

$$\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx$$

$$\int_{-\infty}^0 f(x)dx = \lim_{r\to-\infty}\int_r^0 f(x)dx; \text{ and}$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{\substack{R\to\infty\\ r\to-\infty}}\int_r^R f(x)dx$$

Note limits are taken independently in the final improper integral.

$$\int_{-\infty}^\infty f(x)dx = \lim_{\rho\to\infty}\int_{-\rho}^\rho f(x)dx$$

If the improper integral on the LHS exists then it is equal to the RHS. The RHS may exists even when the improper integral, defined by taking the upper and lower limits independently, does not. In this case we define the Cauchy Principle Value of the integral by:

$$\text{p.v.}\int_{-\infty}^\infty f(x)dx := \lim_{\rho\to\infty}\int_{-\rho}^\rho f(x)dx$$

Jordan Lemma: Let $R = P/Q$ be a rational function, where $\deg(Q) \geq \deg(P) + 1$, and $a \in \mathbb{R}$ be non-zero, then:

$$\lim_{\rho\to\infty}\int_{C_\rho^+}\exp(iaz)\frac{P(z)}{Q(z)}dz = 0 \quad \text{if } a > 0, \text{ and}$$

$$\lim_{\rho\to\infty}\int_{C_\rho^-}\exp(iaz)\frac{P(z)}{Q(z)}dz = 0 \quad \text{if } a < 0$$

where C_ρ^+ and C_ρ^- are the semicircular contours from ρ to $-\rho$ in the upper and lower half-plane respectively.

Improper Integrals with Poles

Infinite Series

Lemma 5.6.4: Let $0 \leq k \leq n$ be non-negative integers, and let $\binom{n}{k}$ be the usual binomial coefficient, and let Γ be a loop with 0 in its interior. Then:

$$\binom{n}{k} = \frac{1}{2\pi i}\int_\Gamma \frac{(1+z)^n}{z^{k+1}}dz$$

Definitions

- **Contour:** A curve Γ from z_0 to z_1 is a contour if it is a finite union of regular curves, which together join z_0 with z_1.
- **Closed:** If its endpoints are the same point. aka $\gamma : [t_0, t_1] \to \mathbb{C}$ satisfies $\gamma(t_0) = \gamma(t_1)$.
- **Simple:** If contour has no self-intersections, except possibly at the endpoints.
- **Loop:** Simple, closed contour.
- **Orientation:** Let Γ be a loop, then we say Γ is positively-orientated if as we move along the curve in the direction of parametrisation, the interior is on the left-hand side.
- **Regular:** if γ is continuously differentiable and $\gamma'(t) \neq 0$ for all $t \in (t_0, t_1)$.
- $(z+\bar{z})/2 = \mathrm{Re}(z)$, and $(z - \bar{z}/2) = \mathrm{Im}(z)$
- $$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}$$
- $$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$
- $$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2}$$
- $$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2}$$
- Antiderivative of $\frac{1}{z(z-1)}$ is $\mathrm{Log}_0(z^{-1} - 1)$.
- Binomial Theorem:
$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k}y^k$$
with
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples

- $\exp(2n\pi i) = 1$
- A continuous function on a closed and bounded set is bounded.
- Real valued and non-constant $\implies$ nowhere holomorphic (Cauchy-Riemann or open mapping theorem).
- Polynomials are holomorphic everywhere.
- $f(z) = \bar{z}$ is nowhere differentiable/holomorphic
- $$f(z) = \frac{z+i}{z-i}$$
is a conformal map that maps the strictly lower plane onto $D_1(0)$
- If Cauchy Riemann equations imply $x_0 = y_0$ then f is differentiable on $\{z \in \mathbb{C} : \mathrm{Re}\, z = \mathrm{Im}\, z\}$, which contains no non-empty open sets, so f is nowhere holomorphic.
- Harmonic on wedge between real and $\frac{\pi}{4}$: $4(x^3y - xy^3) = \mathrm{Im}(z^4)$
- $f(z) = \exp(z)$ is conformal since it's holomorphic and $f'(z) \neq 0$ for all z.
- For a holomorphic function the Taylor series will converge in the largest open disk which does not include any singularity.
- $(1+i)^i = \{\exp(\frac{i}{2}\mathrm{Log}2 - \pi/4 - 2\pi k) | k \in \mathbb{Z}\}$
$= \{e^{-\pi/4-2\pi k}(\cos(\frac{1}{2}\mathrm{Log}2) + i\sin(\frac{1}{2}\mathrm{Log}2)) | k \in \mathbb{Z}\}$